ELEN E3401: Electromagnetics

Spring 2025

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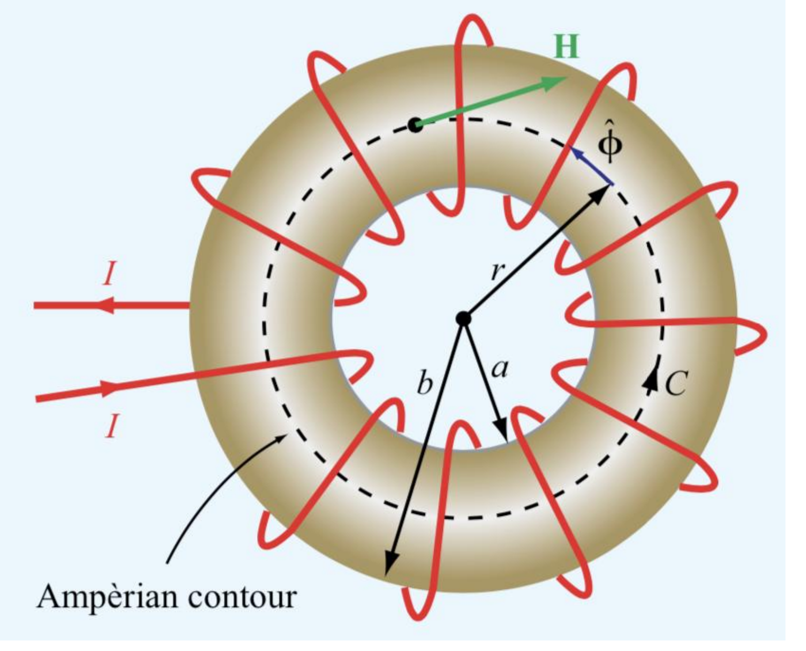
Lecture #16

COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Magnetic Field of Toroid

Applying Ampere's law over contour C

$$\oint_C \vec{H} \cdot d\vec{l} = I$$



Ampere's law: the line integral of $\vec{H}$ around a closed contour C is equal to the current traversing the surface bounded by the contour.

What's the magnetic field outside the toroid?

$$\oint_C \vec{H} \cdot d\vec{l} = \int_0^{2\pi} (-\hat{\varphi}H) \cdot \hat{\varphi} r d\varphi = -2\pi r H = -NI$$

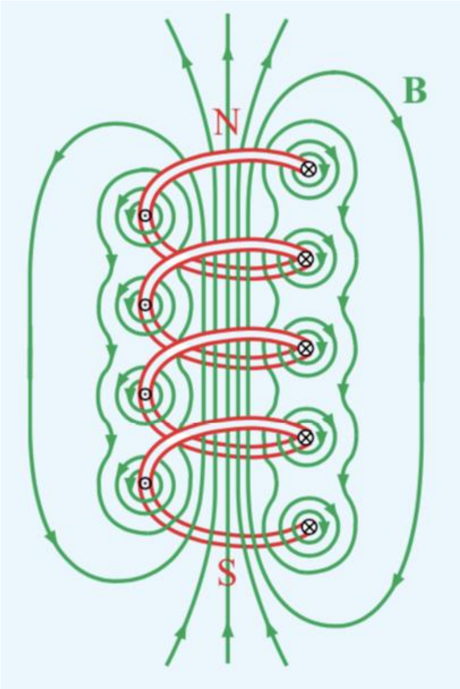
Hence, $H = -\frac{NI}{2\pi r}$ and $\vec{H} = -\hat{\varphi}H = -\hat{\varphi}\frac{NI}{2\pi r}$ for $a < r < b$

Inductance

Inductance → analog of capacitance

Capacitor can store energy in $\vec{E}$ field between conducting surfaces
Inductor stores energy in $\vec{H}$ field near current carrying conductors

<u>Solenoid</u>



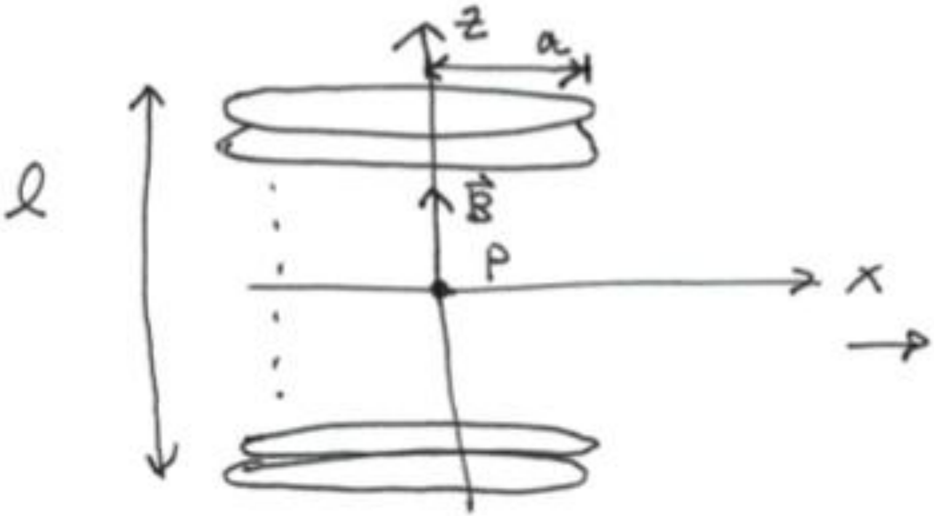
Multiple turns of wire – core can be air or magnetic material, μ

If turns are closely spaced
→ magnetic field can be uniform
→ looks more like permanent magnet

Magnetic Flux in Solenoid

Tightly wound solenoid of length = l
Number of turns / unit length: $n = \frac{N}{l}$

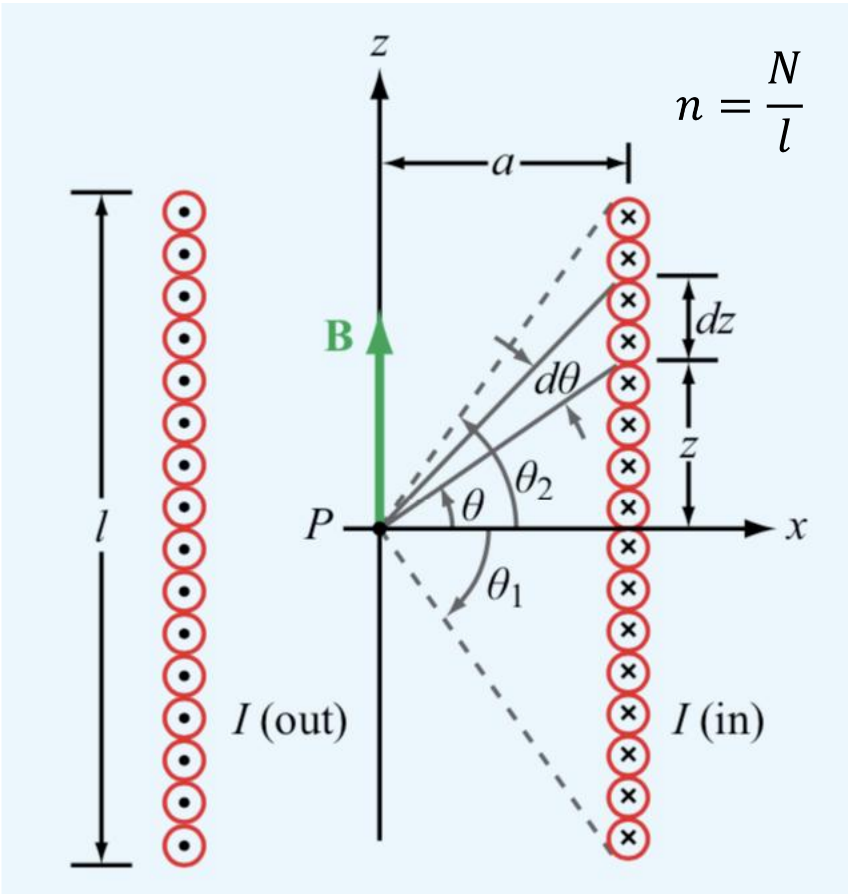
because tight wound, treat helical shape as rings



Radius = a
Length = l

Consider $\vec{B}$ at point P on axis

Solenoid



We previously obtained magnetic field, $\vec{H}$ along axis for 1 loop of current:

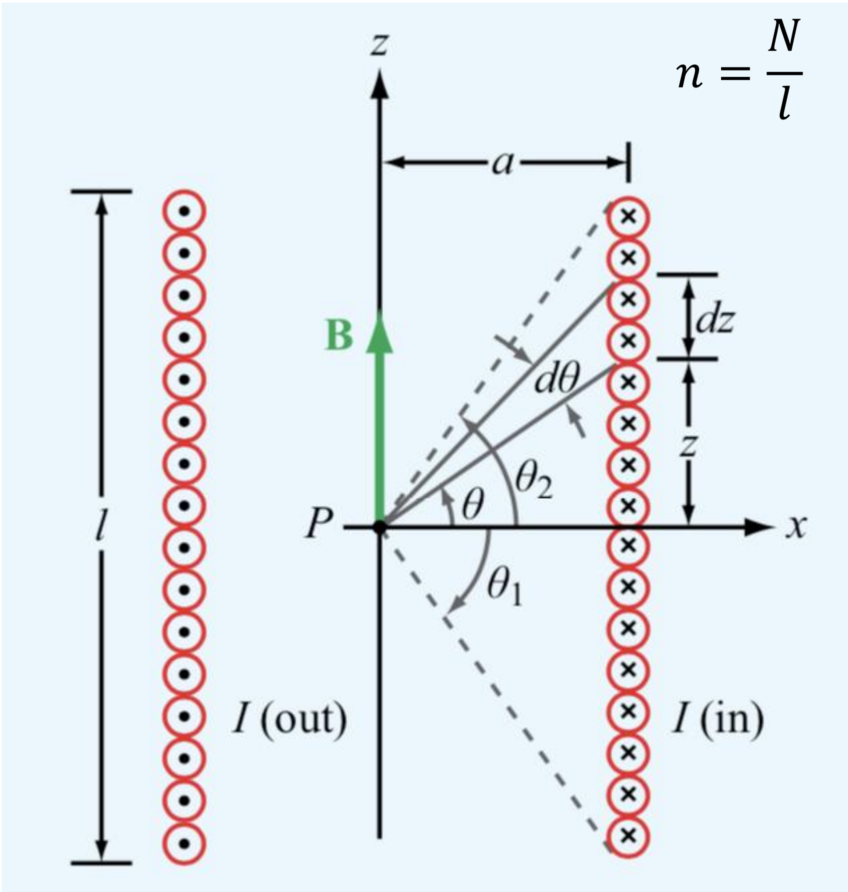
$$\vec{H} = \hat{z}\frac{I'a^2}{2(a^2+z^2)^{3/2}}$$

a = loop radius
I' = current in loop

$$n = \frac{N}{l}$$

For solenoid:
dz of solenoid = equivalent to loop composed of (ndz) turns carrying current
$I' = I(ndz)$

Solenoid



Then, at P:

$$I' = Indz$$

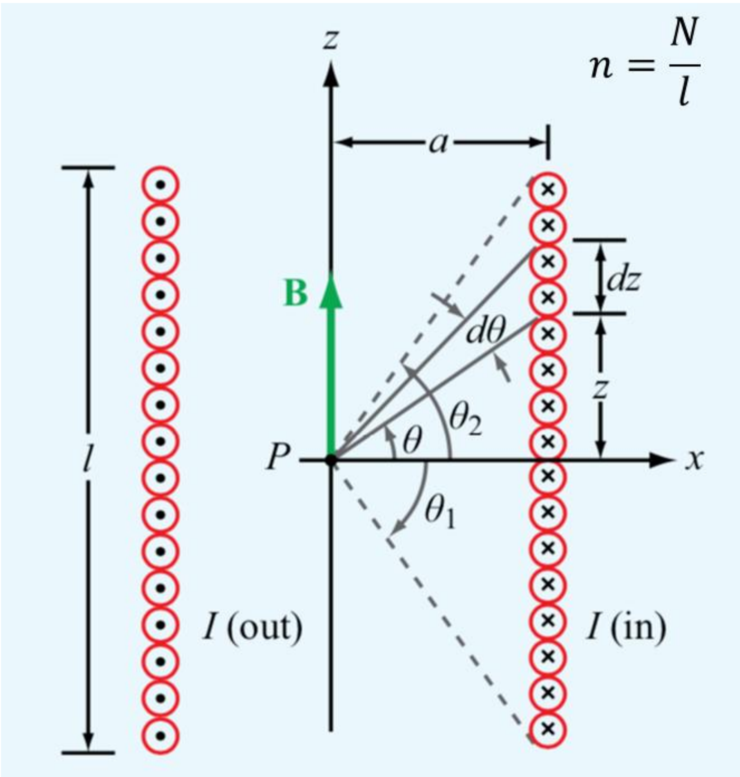
$$d\vec{B} = \mu d\vec{H} = \hat{z}\frac{\mu \overbrace{nI}^{} a^2}{2(a^2+z^2)^{3/2}}dz$$

Total $\vec{B}$: integrate over length of solenoid to compute $\vec{B}$ as seen at P

Angle θ: angle as seen from P of solenoid

$$z = (a)tan\theta \leftarrow tan\theta = \frac{z}{a}$$

Solenoid



$$d\vec{B} = \mu d\vec{H} = \hat{z}\frac{\mu n I a^2}{2(a^2+z^2)^{3/2}}dz$$

$$z = (a)tan\theta$$

$$a^2 + z^2 = \underbrace{a^2 + a^2tan^2(\theta)}_{a^2(1+tan^2(\theta))} = a^2sec^2\theta$$

$$dz = asec^2\theta d\theta$$

$$d\vec{B} = \mu d\vec{H} = \hat{z}\frac{\mu n I a^2}{2(a^2+z^2)^{3/2}}dz$$

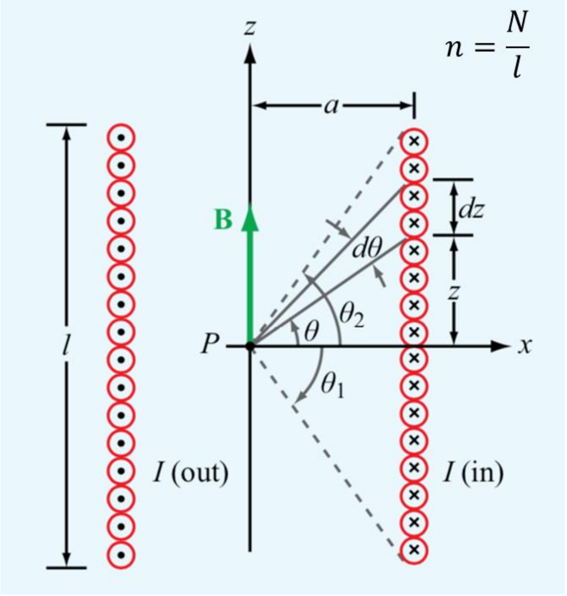
$$\vec{B} = \hat{z}\frac{\mu n I a^2}{2}\int_{\theta_1}^{\theta_2}\frac{asec^2\theta d\theta}{a^3sec^3\theta}$$

$= dz$

$= (a^2+z^2)^{3/2}$

$$\vec{B} = \hat{z}\frac{\mu n I}{2}(sin\theta_2 - sin\theta_1)$$

Solenoid



$$\vec{B} = \hat{z}\frac{\mu n I}{2}(sin\theta_2 - sin\theta_1)$$

If solenoid length $l >> a$, then $\theta_1 = -\theta_2 \approx \pm 90°$ $\theta_1 \approx -90°$ $\theta_2 \approx 90°$

$$\vec{B} \approx \hat{z}\mu n I = \frac{\hat{z}\mu N I}{l}$$ Long solenoid $l\,/a >> 1$

Holds for $\vec{B}$ everywhere inside solenoid (except for ends)

Inductance

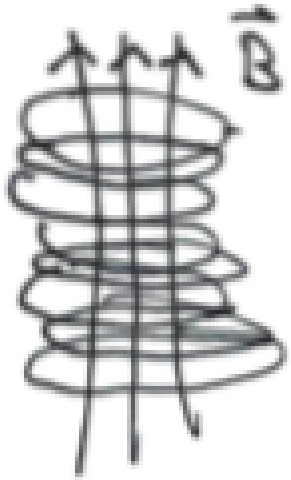
Self inductance: magnetic flux linkage of a coil (or circuit) with itself

Mutual inductance: magnetic flux linkage in a circuit due to magnetic field generated by a current in another circuit

Usually "inductance" refers to self inductance

Magnetic Flux and Magnetic Flux Linkage

Magnetic flux, Φ linking a surface *S* as magnetic flux density passing through surface:



$$\Phi = \int_S \vec{B} \cdot d\vec{s} \;\; [Wb]$$

Solenoid with approximately uniform magnetic field through cross-section:

$$\vec{B} = \frac{\hat{z}\mu NI}{l} \qquad l/a >> 1$$

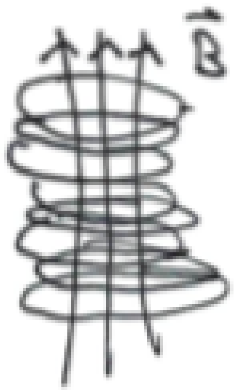
<u>Magnetic flux</u>

Flux linking single loop: $$\Phi = \int_S \overset{\vec{B}}{\frac{\hat{z}\mu NI}{l}} \cdot \overset{d\vec{s}}{\hat{z}ds} = \frac{\mu N}{l} IS$$

Cross section area of loop

Magnetic Flux and Magnetic Flux Linkage

Magnetic flux, Φ linking a surface S as magnetic flux density passing through surface:



$$\Phi = \int_S \vec{B} \cdot d\vec{s} \ [Wb] \qquad \vec{B} = \frac{\hat{z}\mu N I}{l}$$

<u>Magnetic flux</u>

Flux linking single loop: $\Phi = \int_S \frac{\hat{z}\mu N I}{l} \cdot \hat{z} ds = \frac{\mu N}{l} IS$

<u>Magnetic flux linkage</u>: $\Lambda = N\Phi$

<u>Total magnetic flux linking</u> circuit or conducting structure

For solenoid structure: single conductor with multiple loops:

Magnetic Flux Linkage: Λ = flux linking all loops of structure

Solenoid with N turns: $\Lambda = N\Phi = \frac{\mu N^2}{l} IS \quad [Wb]$

Self Inductance

Self-inductance of any conducting structure:

Ratio of: (magnetic flux linkage) / (current flowing through structure)

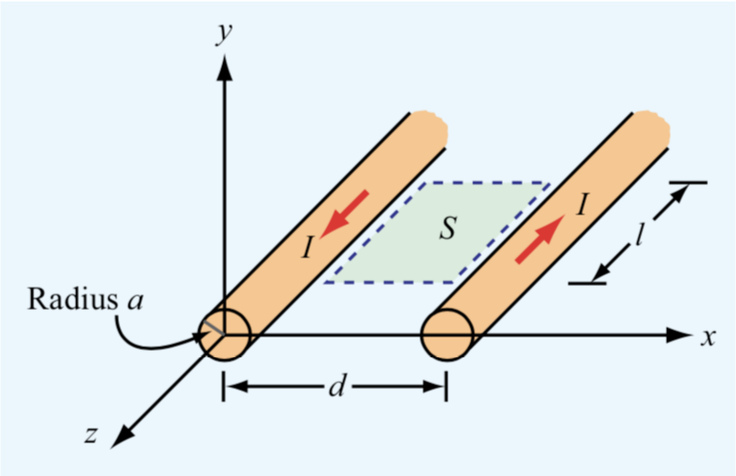
$$L = \frac{\Lambda}{I} \quad [H \; or \; Wb/A]$$

For solenoid: $$L = \frac{\mu N^2}{l} S \;\; [H]$$

For 2-conductor:

$$\Lambda = N\Phi = \Phi \rightarrow N = 1 \; (no \; turns)$$

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I}\int_S \vec{B} \cdot d\vec{s}$$

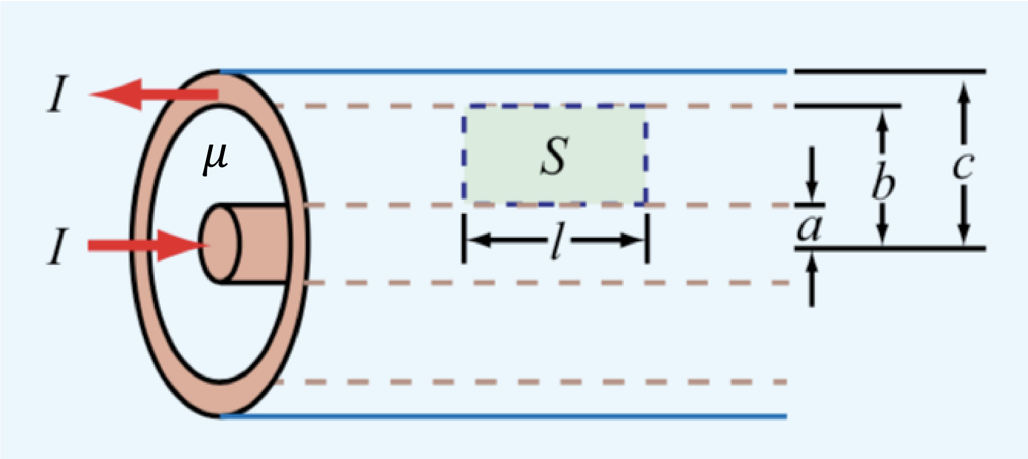


Example: inductance of coax transmission line

Obtain inductance per unit length of a coax transmission line:

Inner/outer conductors of radii a and b

Insulating material of permeability μ



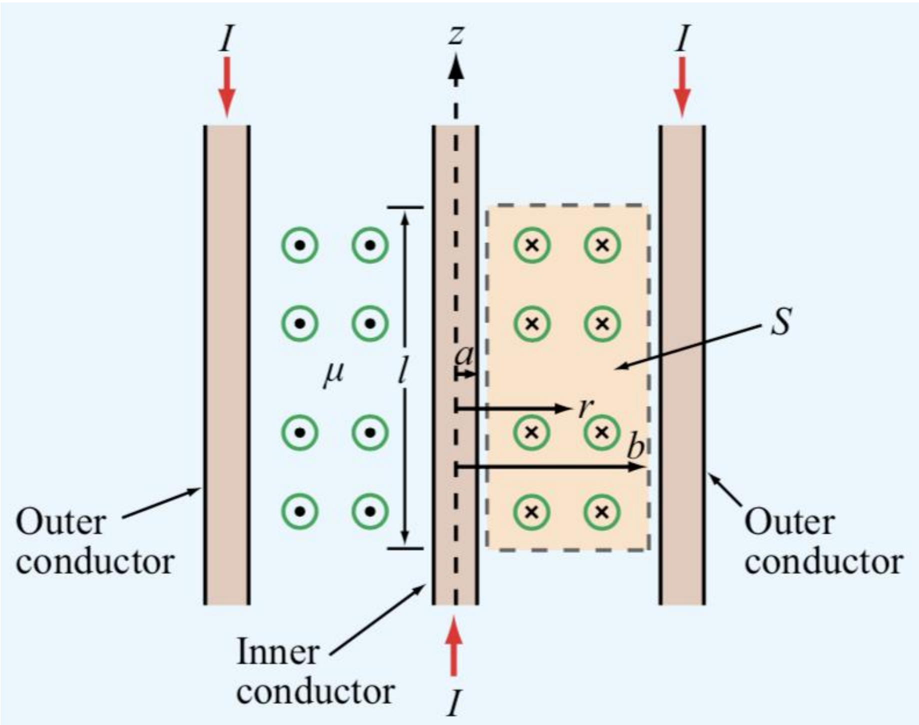
I in inner conductor generates $\vec{B}$ in region between conductors:

$$\vec{B} = \hat{\varphi}\,\frac{\mu I}{2\pi r} \qquad r = \text{radial distance}$$

current is on outside surface of inner conductor and inside surface of outer conductor

Example: inductance of coax transmission line

Cross-section



$$\vec{B} = \hat{\varphi}\frac{\mu I}{2\pi r}$$ r = radial distance

Considering a segment of the TL length l

$\vec{B}$ is perpendicular to S, flux through S:

$$\Phi = l\int_a^b B dr = l\int_a^b \frac{\mu I}{2\pi r} dr$$

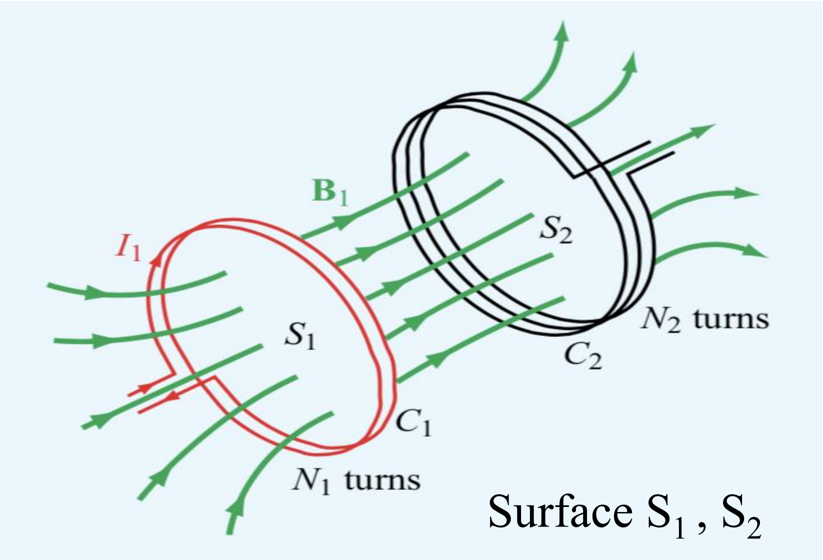
$$\Phi = \frac{\mu I l}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$L' = \frac{inductance}{unit\ length} = \frac{L}{l} = \frac{\Phi}{Il} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$$ Ch.2 Table 2-1

Mutual Inductance

Magnetic coupling between 2 different conducting structures

<u>Consider simple example of 2 multi-turn loops:</u>



I_1 flows in Loop #1, no current in loop #2

$\vec{B}_1$ generated by I_1 → results in flux Φ_{12} through loop #2

$$\Phi_{12} = \int_{S_2} \vec{B}_1 \cdot d\vec{s}$$

If Loop #2 has N_2 turns, all coupled in same way, total magnetic flux linkage:

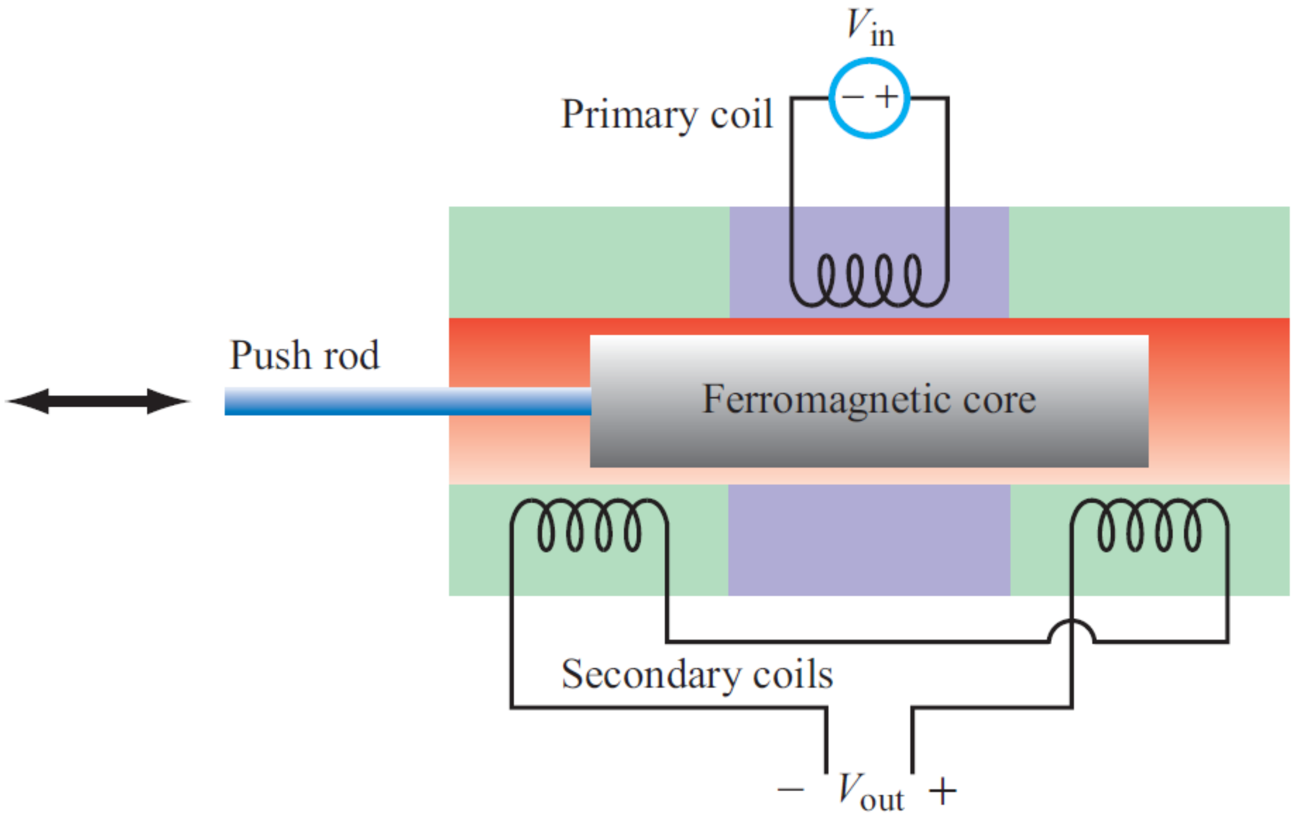
$$\Lambda_{12} = N_2 \Phi_{12} = N_2 \int_{S_2} \vec{B}_1 \cdot d\vec{s}$$

Mutual inductance: $L_{12} = \dfrac{\Lambda_{12}}{I_1} = \dfrac{N_2}{I_1} \displaystyle\int_{S_2} \vec{B}_1 \cdot d\vec{s}$

important for transformers

Inductive Sensors

LVDT (linear variable differential transformer)
can measure displacement with submillimeter precision



Dynamic Fields

Reference	Differential Form	Integral Form	
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}}$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	(6.1)
Faraday's law	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \dfrac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$	(6.2)*
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	(6.3)
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$	(6.4)

*For a stationary surface S.

We will be focusing on Faraday's and Ampère's laws